Lecture 16

Recall. $(M, \mathcal{V})$ CR mfld (hypersurface) w/ $n = $ CR dim M, CR codim $M = 1$. Let θ be real, nonvanishing 1-form s.t. $\forall p \in M$, $\theta(Z_p) = \theta(\bar{Z}_p) = 0$, $\forall Z_p \in \mathcal{V}_p$. The Levi form of M at $p \in M$ is the Hermitian form

$$\mathcal{L}^\theta_p(Z_p, W_p) = \frac{1}{2i}\theta([Z, \bar{W}]), \ \forall Z_p, W_p \in \mathcal{V}_p,$$

where Z, W are local sections of $\mathcal{V}$ extending Z_p, W_p, respectively.

Prop. The Levi form $\mathcal{L}^\theta_p$ does not depend on the extensions Z, W. If $\tilde{\theta} = a\theta$, then $\mathcal{L}^{\tilde{\theta}}_p = a(p)\,\mathcal{L}^\theta_p$.

Pf. The last identity $\mathcal{L}^{\tilde{\theta}}_p = a(p)\,\mathcal{L}^\theta_p$ is obvious from the def. once we prove that $\mathcal{L}^\theta_p$ is well defined (i.e. first part of Prop.)

So, if Z', W' are two other local extensions of Z_p, W_p, then $Z' = Z + cZ''$, where Z'' is another local section and c a function with $c(p) = 0$, and similarly for $W' = W + eW''$. We compute

$$[Z', \overline{W'}] = [Z, \overline{W}] + [Z, \overline{eW''}] + [cZ'', \overline{W}] + [cZ'', \overline{eW''}].$$

Moreover, for any vector fields X, Y and functions u, v, we have

$$\begin{aligned} [uX, vY] &= u(Xv)Y + uvXY - v(Yu)X - uvYX \\ &= uv[X,Y] + u(Xv)Y - v(Yu)X. \end{aligned}$$

Applying this above, using $c(p) = e(p) = 0$ and $\theta(Z) = \theta(\overline{W}) = 0 \Rightarrow \theta([Z', \overline{W'}]) = \theta([Z, \overline{W}])$, i.e., L_p^θ indep. of choice of extension. $\square$

Rem. • The Levi form is not CR invariant since its value at Z_p, W_p depends on your choice of θ. However, you can make it invariant by viewing it as a map $\mathcal{V}_p \times \overline{\mathcal{V}}_p \to \mathbb{C}T_pM/_{\mathcal{V}_p \oplus \overline{\mathcal{V}}}$. It is often more practical to simply extract information about $\mathcal{L}^\theta_p$ that is invariant under the scaling $\mathcal{L}_p \to c\mathcal{L}_p$, $0 \neq c \in \mathbb{R}$.

• In a local frame $Z_1, \dots, Z_n$, the Levi form becomes a Hermitian $n \times n$ matrix $\left(g_{\alpha\bar\beta} = \mathcal{L}^\theta_p(Z_\alpha, \bar Z_\beta)\right)_{\alpha,\beta=1}^n$.

<u>At a given $p \in M$:</u>

- <u>Nondegeneracy</u> ($\det(g_{\alpha\bar\beta}) \neq 0$) is an invariant property.

- Eigenvalues $\lambda_1, \dots, \lambda_n$ are not invariant, but if

$$\begin{cases} l = \min(\#\lambda_i > 0, \#\lambda_i < 0) \\ r = \#\lambda_i = 0 \end{cases}$$

The <u>signature</u> l and <u>corank</u> r are CR invariants.

<u>Def.</u> $(M, \mathcal{V})$ is <u>Levi nondegenerate</u> at p if the Levi form at p is <u>nondegenerate</u> ($\det(g_{\alpha\bar\beta}) \neq 0$, or, eq., $r = 0$). It is <u>strictly pseudoconvex</u> at p if the Levi form is (pos. or neg.) definite ($r=0$, $l=0$) at p.

<u>Real hypersurfaces in $\mathbb{C}^n$.</u>

Let now $(M, \mathcal{V})$ be an embedded real hypersurface $M \subseteq \mathbb{C}^{n+1}$, where $\mathcal{V} = T^{1,0}M$. Let $\rho = 0$ be a local defining equation. Thus, CR dim $M = n$. If we let $e: M \to \mathbb{C}^{n+1}$ be the embedding, then we may take as $\theta = i e^* \partial\rho$. <u>Check</u>:

① θ is real, b/c $0 = e^* d\rho = e^*\partial\rho + \underbrace{e^*\bar\partial\rho}_{\overline{e^*\partial\rho}}$

$\Rightarrow$ $e^*\partial\rho$ is purely imag. $\Rightarrow$ $i e^*\partial\rho$ is real.

② $Z = \sum_{j=1}^{n+1} \xi^{pj} \frac{\partial}{\partial z_j}$ is in $T_p^{1,0}M \Leftrightarrow$

$$0 = \sum_{j=1}^{n+1} \frac{\partial \rho}{\partial z_j}(p)\xi^{pj} = \partial\rho|_p(Z). \Rightarrow$$

$\theta = i\, e^* \partial\rho$ annihilates $T_p^{1,0}M$ and since ρ is real also then $T_p^{0,1}M$.

③ $\underline{\theta \neq 0}$.

Since $d\rho^{\perp} = TM$, let $T_p = \sum\left(\eta_j \frac{\partial}{\partial z_j} + \overline{\eta_j}\frac{\partial}{\partial \overline{z_j}}\right) \neq 0$ be in $d\rho^{\perp} \Leftrightarrow 0 = d\rho(T_p) = \sum_{j=1}^{n}\left(\frac{\partial \rho}{\partial z_j}\eta_j + \frac{\partial \rho}{\partial \overline{z_j}}\overline{\eta_j}\right)$
$\Rightarrow \sum \frac{\partial \rho}{\partial z_j}\eta_j = -\sum \frac{\partial \rho}{\partial \overline{z_j}}\overline{\eta_j}$. If we further choose T_p s.t. $T_p \in T_pM \setminus H_p$, then $\sum \frac{\partial \rho}{\partial z_j}\eta_j \neq 0$.
Now $\theta(T_p) = i\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p)\,\eta_j \neq 0 \Rightarrow \theta \neq 0$.